Lower bound for sum of squares of ratios altitudes to sidelengths
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Let $a,b,$ and c be the lengths of the sides opposite vertices $A,B,$ and $C,$ respectively, a nonobtuse triangle. Let $h_a,h_b,$ and h_c be the corresponding lengths of the altitudes.
Show that: $\left(\frac{h_a}{a}\right)^2+\left(\frac{h_b}{b}\right)^2+\left(\frac{h_c}{c}\right)^2 \geq \frac{9}{4}.$
Solution by Arkady Alt, **San Jose** ,**California**, **USA**.
First we will prove that in any triangle with sidelengths a,b,c holds inequality
(**1**) $\Delta(a,b,c)\cdot\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\geq 9,$ where $\Delta(a,b,c) := 2\sum ab-\sum a^2.$
Proof.
Let $x := s-a, y := s-b, z := s-c$ where s is semiperimeter of the triangle and let $p := \sum xy, q := xyz.$ Then $x,y,z>0$ and assuming $s=1$ (due homogeneity of (**1**)) we obtain $x+y+z=1, a=1-x, b=1-y, c=1-z, \sum a=2, \sum ab=$
$\sum(1-x)(1-y)=\sum(z+xy)=1+p, \sum a^2=\left(\sum a\right)^2-2\sum ab=2(1-p),$
$abc=(1-x)(1-y)(1-z)=p-q,\ \sum a^2b^2=\left(\sum ab\right)^2-2abc\sum a=$
$(1+p)^2-4(p-q)=(1-p)^2+4q, \Delta(a,b,c)=4\left(\sum ab\right)^2-\left(\sum a\right)^2=4p$ and
inequality (**1**) becomes $\dfrac{4p\left((1-p)^2+4q\right)}{(p-q)^2}\geq 9.$
Since $3p=3\sum xy\leq\left(\sum x\right)^2=1$ and $9q\geq 4p-1$ (normalized by $\sum x=1$ Schure's inequality $\sum x(x-y)(x-z)\geq 0$ in p,q notation) then $q\geq\dfrac{4p-1}{9}$
and noting that $\dfrac{(1-p)^2+4q}{(p-q)^2}$ increases by $q\in(0,p/9]$ $(9q=9xyz\leq$ $\left(\sum x\right)\cdot\left(\sum xy\right)=p)$ we obtain for $p\in(1/4,1/3]$ that
$$\frac{4p\left((1-p)^2+4q\right)}{(p-q)^2}-9\geq\frac{4p\left((1-p)^2+4\cdot\frac{4p-1}{9}\right)}{\left(p-\frac{4p-1}{9}\right)^2}-9=\frac{9(4p-1)(1-3p)^2}{(5p+1)^2}\geq 0.$$
If $p\in(0,1/4]$ then $\dfrac{4p\left((1-p)^2+4q\right)}{(p-q)^2}-9>\dfrac{4p\left((1-p)^2+4\cdot 0\right)}{(p-0)^2}=\dfrac{(4-p)(1-4p)}{p}>0.$
Thus, equality occurs iff $p=1/3$ and $q=\dfrac{4\cdot(1/3)-1}{9}=\dfrac{1}{27}\Leftrightarrow x=y=z=1/3$
that is in original notation iff $a=b=c.$ ■
Coming back to the original problem in case of acute triangle, by replacing (a,b,c) in inequality (**1**) with (a^2,b^2,c^2) we obtain, since $\Delta(a^2,b^2,c^2)=16F^2$, where F is area of the triangle, that $16F^2\cdot\sum\frac{1}{a^4}\geq 9\Leftrightarrow\sum\frac{4a^2h_a^2}{a^4}\geq\frac{9}{4}\Leftrightarrow\sum\frac{h_a^2}{a^2}\geq\frac{9}{4}.$
In the case $\triangle ABC$ is right angled with $C=90^\circ,$ we have
$c^2=a^2+b^2, h_a=b, h_b=a, h_c=\frac{ab}{c}$
and $\left(\frac{h_a}{a}\right)^2+\left(\frac{h_b}{b}\right)^2+\left(\frac{h_c}{c}\right)^2=\left(\frac{b}{a}\right)^2+\left(\frac{a}{b}\right)^2+\left(\frac{ab}{a^2+b^2}\right)^2=\frac{a^4+b^4}{a^2b^2}+\frac{a^2b^2}{(a^2+b^2)^2}.$

Since $(a^2+b^2)^2 \le 2(a^4+b^4)$ then, denoting $t := \dfrac{a^4+b^4}{a^2b^2} \ge 2$ we obtain that $\dfrac{a^4+b^4}{a^2b^2} + \dfrac{a^2b^2}{(a^2+b^2)^2} \ge t + \dfrac{1}{2t} \ge \dfrac{9}{4}$, because $t + \dfrac{1}{2t} - \dfrac{9}{4} = \dfrac{(4t-1)(t-2)}{4t} \ge 0$.

Thus, in inequality of the problem equality occurs iff the triangle is equilateral or isosceles right angled.